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Published in:
Nuclear Physics B

DOI:
[10.1016/0550-3213\(96\)00120-4](https://doi.org/10.1016/0550-3213(96)00120-4)

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
1996

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Behrndt, K., Bergshoeff, E., & Janssen, B. (1996). Type II duality symmetries in six dimensions. *Nuclear Physics B*, 467(1), 100 - 126. [https://doi.org/10.1016/0550-3213\(96\)00120-4](https://doi.org/10.1016/0550-3213(96)00120-4)

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Nuclear Physics B 467 (1996) 100–126

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PHYSICS B

Type II duality symmetries in six dimensions

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Received 28 December 1995; accepted 1 March 1996

Abstract

We discuss the different discrete duality symmetries in six dimensions that act within and between (i) the 10-dimensional heterotic string compactified on T^4 , (ii) the 10-dimensional type IIA string compactified on $K3$ and (iii) the 10-dimensional type IIB string compactified on $K3$. In particular we show that the underlying group-theoretical structure of these discrete duality symmetries is determined by the proper cubic group C/\mathbb{Z}_2 . Our group theoretical interpretation leads to simple rules for constructing the explicit form of the different discrete type II duality symmetries in an arbitrary background. The explicit duality rules we obtain are applied to construct dual versions of the 6-dimensional chiral null model.

1. Introduction

Duality symmetries are playing an increasingly important role in relating different types of string theories and in investigating their non-perturbative behaviour. In many cases the duality symmetries manifest themselves at the level of the low-energy effective action as solution-generating transformations. Having the application as solution-generating transformation in mind it is important to have knowledge of the explicit form of the duality symmetry in an arbitrary curved background. For the 10-dimensional heterotic string the so-called T -duality rule has been given some time ago by Buscher [1]. This rule includes the well-known $R \rightarrow 1/R$ duality for the special background $\mathcal{M}_9 \times T^1$, where \mathcal{M}_9 is a 9-dimensional Minkowski space and T^1 a circle of radius R . For the 10-dimensional type II strings it has been shown that the T duality for the background $\mathcal{M}_9 \times T^1$ provides a map between the type IIA and type IIB superstring [2,3]. The corresponding expression in a general curved background has been given recently in [4] thereby generalizing Buscher's rules to the type II case. Both the type I

as well as the type II 10-dimensional duality rules have been applied to generate new solutions to the ten-dimensional low-energy string effective action.

In six dimensions an important role is played by the so-called string/string duality [5]. This string/string duality relates the weakly coupled $D = 10$ heterotic string compactified on T^4 to the strongly coupled type IIA superstring compactified on $K3$ and vice versa [6–9]. Additional 6-dimensional dualities have been considered involving the $D = 10$ type IIB superstring compactified on $K3$ [7,10,11], leading to the concept of string/string/string triality [10]. It is the purpose of this paper to explain the group-theoretical structure underlying the different discrete 6-dimensional dualities. Our analysis will lead to a simple do-it-yourself kit for constructing the explicit form of the discrete duality symmetries that act within and between (i) the 10-dimensional heterotic string compactified on T^4 , (ii) the 10-dimensional type IIA string compactified on $K3$ and (iii) the 10-dimensional type IIB string compactified on $K3$.

To explain the main idea of this paper it is instructive to analyse the much simpler group theoretical structure underlying the $D = 10$ duality symmetries. The main purpose of this $D = 10$ analysis is not to present new results but to exemplify the more complicated situation of the duality symmetries in six dimensions which is the topic of this paper.

We first consider the $D = 10$ heterotic string. The bosonic background fields are a metric \hat{g} , an antisymmetric tensor \hat{B} and a dilaton $\hat{\phi}$. The dilaton is related to the string coupling constant g_s via $g_s = e^{\hat{\phi}}$. The zero-slope limit action in the string-frame metric is given by¹

$$I_{\text{heterotic}} = \frac{1}{2} \int d^{10}x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left\{ -\hat{R} + 4|d\hat{\phi}|^2 - \frac{3}{4}|d\hat{B}|^2 \right\}. \quad (1)$$

We assume that the background fields have an isometry in a given, let us say, \underline{x} direction. The T -duality rules are easiest formulated by rewriting the 10-dimensional action in nine dimensions via the process of dimensional reduction [13,14]. Besides a 9-dimensional metric, antisymmetric tensor and dilaton the dimensionally reduced theory contains two additional vectors A and B and one scalar σ . The vectors A and B originate from the 10-dimensional metric and antisymmetric tensor, respectively. The string coupling constant g_s and the compactification radius R are related to the 9-dimensional dilaton ϕ and the scalar σ via

$$R = e^{\sigma/2}, \quad g_s = e^{\phi + \sigma/4}. \quad (2)$$

To understand how the different discrete duality symmetries are realized in nine dimensions it is enough to consider the kinetic terms of the two vector fields A and B

¹ We use the notation and conventions of [4,12]. In particular, fields are hatted before and unhatted after dimensional reduction. Throughout this paper we will use a form notation. If A is a p -form and B a q -form, then $|A|^2$ means $A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p}$, AB means $A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]}$ and dA means $\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$.

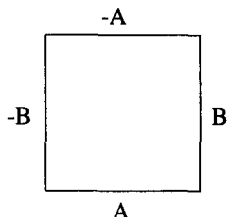


Fig. 1. Each discrete symmetry of the square corresponds to a symmetry acting on the two vectors A and B . The four sides of the square correspond to the pairs $(A, -A)$ and $(B, -B)$.

and their coupling to σ and ϕ which we write in the following suggestive form, using the Einstein-frame metric²

$$I_{AB\sigma} = \frac{1}{2} \int d^9x \sqrt{g} \left[e^{\mathcal{Q}_A \cdot \Phi} |dA|^2 + e^{\mathcal{Q}_B \cdot \Phi} |dB|^2 \right], \quad (3)$$

where $\Phi = (\sigma, \phi)$ and

$$\mathcal{Q}_A = \left(1, -\frac{4}{7}\right), \quad \mathcal{Q}_B = \left(-1, -\frac{4}{7}\right). \quad (4)$$

It is not difficult to analyze the discrete duality symmetries of this action. It turns out that on the two vectors A and B one can realize the 8-element finite dihedral group D_4 [12]. The easiest way to see how this group is realized is to write a square, like in Fig. 1, with sides $(A, -A)$ and $(B, -B)$. Every discrete symmetry of the square corresponds to a symmetry acting on the two vectors. For instance, the 2-order element of D_4 given by a reflection around the diagonal from the left upper corner to the right lower corner corresponds to the transformation $A' = B, B' = A$. As another example we mention the 4-order element of D_4 given by a (counterclockwise) rotation over 90 degrees. It corresponds to the symmetry $A' = B, B' = -A$.

To see which discrete group is realized on σ , we write the two vectors \mathcal{Q}_A and \mathcal{Q}_B as the corners of a 1-dimensional line-segment, like in Fig. 2. One thus finds that the 2-element discrete group

$$\mathbb{Z}_2 = D_4 / (\mathbb{Z}_2 \times \mathbb{Z}_2) \quad (5)$$

is realized on the scalar σ , i.e. to every four symmetries acting on the vectors corresponds a single symmetry acting on the scalar. The action of \mathbb{Z}_2 on σ, ϕ is given by

$$\begin{aligned} \mathbf{e}: \quad \sigma' &= \sigma, & \phi' &= \phi, \\ \mathbf{T}: \quad \sigma' &= -\sigma, & \phi' &= \phi. \end{aligned} \quad (6)$$

One may alternatively describe this \mathbb{Z}_2 duality by its action on the compactification radius R and the string coupling constant g_s as follows:

² Actually, the situation is a bit more complicated than described below since also the 9-dimensional anti-symmetric tensor is involved in the discrete duality symmetries. For more details, see Ref. [12].

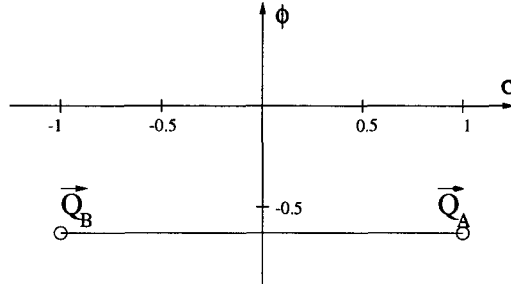


Fig. 2. Each symmetry of the line-segment corresponds to a symmetry acting on σ, ϕ . The two corners of the line-element are given by the two vectors Q_A and Q_B defined in Eq. (4).

$$\begin{aligned} \mathbf{e}: \quad R' &= R, & g'_s &= g_s, \\ \mathbf{T}: \quad R' &= \frac{1}{R}, & g'_s &= \frac{g_s}{R}. \end{aligned} \quad (7)$$

To each element of \mathbb{Z}_2 corresponds four elements of D_4 acting on the two vectors. The specific transformations of the vectors are given by³

$$\begin{aligned} \mathbf{e}: \quad A' &= A, & B' &= B, \\ \mathbf{T}: \quad A' &= B, & B' &= A. \end{aligned} \quad (8)$$

Combining the above $D = 9$ discrete symmetries with the dimensional reduction formulae relating the $D = 9$ fields to the $D = 10$ ones, it is now straightforward to construct the $D = 10$ form of the duality rules. This leads basically to Buscher's duality rules. All the other discrete dualities differ from the unit transformation or Buscher's rules by some additional sign changes in the 10-dimensional analogues of the 9-dimensional vectors. In practice, it often means that the sign of some charges in a given solution are undetermined.

We now extend the above analysis to the $D = 10$ type II theories. Both the type IIA and type IIB theories contain the action (1) as a common subsector. One finds that after dimensional reduction the D_4 symmetry does not survive as a symmetry of the full $D = 9$ type II action but is broken according to⁴

$$D_4 \rightarrow \mathbb{Z}_2. \quad (9)$$

In particular, it turns out that the \mathbb{Z}_2 T -duality acting on the scalar σ is broken. This does *not* mean that there is no type II T -duality. It just means that from the 10-dimensional point of view the T -duality describes a map between *different* theories. In this case the T -duality establishes a map between the type IIA and type IIB superstring [2,3], as indicated in Fig. 3.

³ We only give two elements of D_4 . To every element below one can associate three more elements by changing (in three possible ways) signs in the given transformation rules.

⁴ The unbroken \mathbb{Z}_2 transformation acts on the two vectors A and B as $A' = -A$ and $B' = -B$ [4,12].

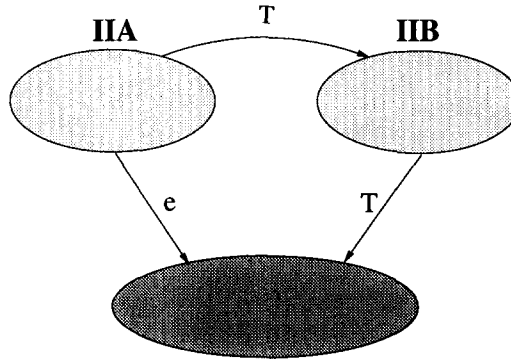


Fig. 3. The type II T -duality in ten dimensions describes a map between the type IIA and type IIB superstring. The reduction to $D = 9$ of the type IIA (type IIB) theory is indicated in the figure with e (T).

We call the reduction formula that maps the 10-dimensional type IIA and IIB theory onto *the same* 9-dimensional theory, e and T , respectively.⁵ The reason for this is that, when restricted to the common subsector given in (1), the IIB reduction formula becomes the T -dual of the IIA reduction formula. For instance, for the special background, $\mathcal{M}_9 \times T^1$, when the IIA theory is dimensionally reduced over a circle with radius R_A , the type IIB theory is dimensionally reduced over a circle with radius $1/R_B$. Using this suggestive notation, it is straightforward to construct the explicit form of the $D = 10$ T -duality rule that establishes the map from the type IIA onto the type IIB theory. It is just a composition of the two reduction formulae indicated in Fig. 3. For instance, the map from IIA to IIB is obtained by first reducing IIA with e and then oxidizing back to IIB with T^{-1} . Similarly, the map from IIB to IIA is obtained by first reducing IIB with T and then oxidizing back to IIA with e^{-1} :

$$\begin{aligned} T(IIA \rightarrow IIB) &= T^{-1} \times e = T \times e = T, \\ T(IIB \rightarrow IIA) &= e^{-1} \times T = e \times T = T. \end{aligned} \quad (10)$$

Note that the $D = 10$ type II dualities, although not being the symmetry of a *single* theory, still satisfy the \mathbb{Z}_2 group structure in the sense that

$$\begin{aligned} T(IIB \rightarrow IIA) \times T(IIA \rightarrow IIB) &= \mathbb{I}(IIA \rightarrow IIA), \\ T(IIA \rightarrow IIB) \times T(IIB \rightarrow IIA) &= \mathbb{I}(IIB \rightarrow IIB). \end{aligned} \quad (11)$$

This is due to our notation of the reduction formulae which is such that, when restricted to the common subsector, each reduction formula (and its inverse) is in one-to-one correspondence with a specific \mathbb{Z}_2 symmetry of the $D = 10$ heterotic action (1) after dimensional reduction to $D = 9$.

It is the purpose of this paper to show that the above analysis in $D = 9, 10$ can be repeated for the more complicated case in $D = 5, 6$. Basically, apart from being

⁵ The corresponding inverse reduction formulae, or decompactification formulae, are indicated by e^{-1} and T^{-1} , respectively.

Table 1

Discrete duality groups in $D = 9, 10$ and $D = 5, 6$

$D = 9, 10$	$D = 5, 6$
D_4 \mathbb{Z}_2	C/\mathbb{Z}_2 D_3

technically more complicated, the only difference with the above analysis is that different discrete duality groups are involved. To be precise, compared with the $D = 9, 10$ situation we will be dealing in $D = 5, 6$ with the groups indicated in Table 1.

Here C/\mathbb{Z}_2 is the 24-element proper cubic group and D_3 is the 6-element dihedral group. The analogues of Figs. 1–3 are given in Figs. 4–6, respectively. In particular Fig. 6 will provide us with simple and elegant rules for constructing the explicit form of all discrete duality symmetries indicated in the figure. This will turn out to be extremely useful for the purpose of constructing new solutions corresponding to the different string effective actions involved.

The organization of this paper is as follows. In Section 2 we discuss the common sector in six dimensions and exhibit its discrete duality symmetries. In Section 3 we give and discuss the explicit form of the zero-slope limit effective actions corresponding to (i) the 10-dimensional heterotic string compactified on T^4 , (ii) the 10-dimensional type IIA string compactified on $K3$ and (iii) the 10-dimensional type IIB string compactified on $K3$. The dimensional reduction of these effective actions onto the *same* 5-dimensional type II theory is discussed in Section 4. In Section 5 we show how, using the results of Section 4, one may construct in a simple way the explicit form of the different 6-dimensional duality transformations. Finally, in Section 6 these duality rules are applied to construct dual versions of the 6-dimensional chiral null model.

2. The common sector

Each of the $D = 6$ string theories discussed in this paper (heterotic, type IIA, type IIB) contains as a common subsector a metric \hat{g} , an antisymmetric tensor \hat{B} and a dilaton $\hat{\phi}$. These fields define the common sector of each string theory. In this section we will first investigate the duality symmetries of this common sector.⁶ The action for the common sector in the string-frame metric is given by

$$I_{\text{common}} = \frac{1}{2} \int d^6x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left\{ -\hat{R} + 4|d\hat{\phi}|^2 - \frac{3}{4}|\hat{d}\hat{B}|^2 \right\}. \quad (12)$$

The special thing about six dimensions is that the equations of motion corresponding to the common sector are invariant under so-called string/string duality transformations [5]. These transformations are easiest formulated in the (6-dimensional) Einstein-frame metric

⁶ The contents of this section has some overlap with Ref. [15].

$$\hat{g}_E = e^{-\hat{\phi}} \hat{g}_S, \quad (13)$$

which is invariant under the string/string duality. The action for the common sector in the Einstein-frame metric is given by⁷

$$I_{\text{common}} = \frac{1}{2} \int d^6 x \sqrt{-\hat{g}} \left\{ -\hat{R} - |d\hat{\phi}|^2 - \frac{3}{4} e^{-2\hat{\phi}} |d\hat{B}|^2 \right\}. \quad (14)$$

The string/string duality rules in the Einstein frame are given by

$$\hat{\phi}' = -\hat{\phi}, \quad (d\hat{B})' = e^{-2\hat{\phi}} {}^*(d\hat{B}), \quad (15)$$

where

$${}^*(d\hat{B})_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv \frac{1}{3! \sqrt{-\hat{g}}} \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}\hat{\sigma}\hat{\tau}} (d\hat{B})^{\hat{\lambda}\hat{\sigma}\hat{\tau}}. \quad (16)$$

We now discuss the reduction to five dimensions, assuming there is an isometry in a given, let us say, \underline{x} -direction. Using both in $D = 6$ as well as $D = 5$ the string-frame metric, the 6-dimensional fields are expressed in terms of the five-dimensional ones as follows⁸:

$$\begin{aligned} \hat{g}_{\underline{x}\underline{x}} &= -e^{-4\sigma/\sqrt{3}}, \\ \hat{g}_{\underline{x}\mu} &= -e^{-4\sigma/\sqrt{3}} A_\mu, \\ \hat{g}_{\mu\nu} &= g_{\mu\nu} - e^{-4\sigma/\sqrt{3}} A_\mu A_\nu, \\ \hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu} B_{\nu]}, \\ \hat{B}_{\underline{x}\mu} &= B_\mu, \\ \hat{\phi} &= \phi - \frac{1}{\sqrt{3}} \sigma. \end{aligned} \quad (17)$$

The dimensionally reduced action in the (5-dimensional) string frame metric is given by

$$\begin{aligned} I_{\text{common}} &= \frac{1}{2} \int d^5 x \sqrt{g} e^{-2\phi} \left[-R + 4|d\phi|^2 - \frac{3}{4}|H|^2 - \frac{4}{3}|d\sigma|^2 \right. \\ &\quad \left. + e^{-4\sigma/\sqrt{3}} |dA|^2 + e^{4\sigma/\sqrt{3}} |dB|^2 \right], \end{aligned} \quad (18)$$

with

$$H = dB + AdB + BdA. \quad (19)$$

We next use the fact that five dimensions is special in the sense that in this dimension the antisymmetric tensor B is dual to a vector C [7]. In terms of this vector C the action is given by⁹

⁷ We will denote the string-frame and Einstein-frame metric with the same symbol. Whenever confusion could arise, we will explicitly denote with a subindex whether a metric is in Einstein or string frame.

⁸ We have used the decomposition $\hat{\mu} = (\mu, \underline{x})$.

⁹ For clarity we give here the component form of the last (topological) term in the action:

$$I_{\text{common}} = \frac{1}{2} \int_{\mathcal{M}_5} d^5x \sqrt{g} e^{-2\phi} \left[-R + 4|d\phi|^2 - \frac{4}{3}|d\sigma|^2 + e^{-4\sigma/\sqrt{3}}|dA|^2 + e^{4\sigma/\sqrt{3}}|dB|^2 + e^{4\phi}|dC|^2 \right] - \frac{1}{2} \int_{\mathcal{M}_6} dA dB dC, \quad (21)$$

where \mathcal{M}_6 is a 6-manifold with boundary \mathcal{M}_5 .

To study the symmetries of the dimensionally reduced action it is convenient to use the (5-dimensional) Einstein frame metric

$$g_E = e^{-4\phi/3} g_S, \quad (22)$$

in terms of which the action becomes

$$I_{\text{common}} = \frac{1}{2} \int_{\mathcal{M}_5} d^5x \sqrt{g} \left[-R - \frac{4}{3}|d\phi|^2 - \frac{4}{3}|d\sigma|^2 + e^{-4Q_A \cdot \Phi/3}|dA|^2 + e^{-4Q_B \cdot \Phi/3}|dB|^2 + e^{-4Q_C \cdot \Phi/3}|dC|^2 \right] - \frac{1}{2} \int_{\mathcal{M}_6} dA dB dC, \quad (23)$$

where $\Phi = (\sigma, \phi)$ and

$$\begin{aligned} Q_A &= (\sqrt{3}, 1), \\ Q_B &= (-\sqrt{3}, 1), \\ Q_C &= (0, -2). \end{aligned} \quad (24)$$

Given the above form of the dimensionally reduced action it is not difficult to analyse its discrete duality symmetries. It turns out that on the three vectors one can realize the 24-element finite group C/\mathbb{Z}_2 where C is the so-called cubic group. The easiest way to see how this group is realized is to write a cube, like in Fig. 4, with faces $(A, -A)$, $(B, -B)$ and $(C, -C)$.

The reason that we only consider the 24 proper symmetries and not the full 48-element cubic group is that only the proper elements leave the last (topological) term in the action (23) invariant. The proper cubic group has elements of order 2 and 3.¹⁰ An example of a 2-order element is the reflection around the diagonal vertical plane that connects the right-front of the cube to the left-back of the cube. An example of a 3-order element is given by a (counter-clockwise) rotation of 120 degrees with axis

$$-\frac{1}{2} \int_{\mathcal{M}_6} dA dB dC = -\frac{1}{2} \int_{\mathcal{M}_6} d^6x \epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}\hat{\sigma}\hat{\tau}} \partial_{\hat{\mu}} A_{\hat{\nu}} \partial_{\hat{\rho}} B_{\hat{\lambda}} \partial_{\hat{\sigma}} C_{\hat{\tau}} = -\frac{1}{2} \int_{\mathcal{M}_5} d^5x \epsilon^{\mu\nu\rho\lambda\sigma} A_{\mu} \partial_{\nu} B_{\rho} \partial_{\lambda} C_{\sigma}. \quad (20)$$

¹⁰ The full cubic group also has elements of order 4. An example of such a 4-order element is a rotation of 90 degrees with axis the line going from the centre of the lower face to the centre of the upper face of the cube.

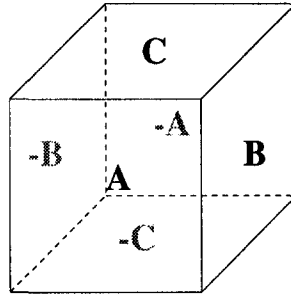


Fig. 4. Each proper discrete symmetry of the cube corresponds to a symmetry acting on the three vectors. The six faces of the cube correspond to the pairs $(A, -A)$, $(B, -B)$ and $(C, -C)$.

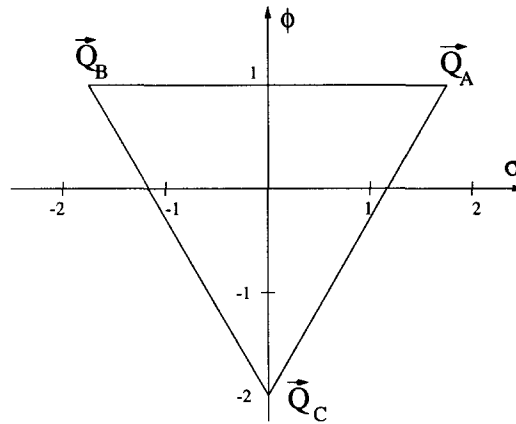


Fig. 5. Each symmetry of the equilateral triangle corresponds to a symmetry acting on the two scalars. The three corners of the triangle are given by the three vectors Q_A , Q_B and Q_C defined in Eq. (24).

the line going from the upper right corner at the front to the lower left corner at the back of the cube. Each of the 24 proper discrete symmetries of the cube naturally leads to a discrete symmetry acting on the three vectors. For instance, the 2- and 3-order elements given above induce the following discrete symmetries acting on the vectors, respectively:

$$\begin{aligned} A' &= B, & B' &= A, & C' &= C, \\ A' &= B, & B' &= C, & C' &= A. \end{aligned} \quad (25)$$

To see which discrete group is realized on the two scalars, it is easiest to write the three vectors Q_A , Q_B and Q_C as the corners of an equilateral triangle, like in Fig. 5. It was pointed out by Kaloper [15] that on the scalars one can realize the 6-element dihedral group

$$D_3 = C / (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2), \quad (26)$$

i.e. to every four symmetries acting on the vectors one relates a single symmetry acting on the scalars. The action of the six elements of D_3 on the scalars is given by¹¹

$$\begin{aligned}
 \mathbf{e}: \quad & \sigma' = \sigma, \\
 & \phi' = \phi, \\
 \mathbf{T}: \quad & \sigma' = -\sigma, \\
 & \phi' = \phi, \\
 \mathbf{S}: \quad & \sigma' = \frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi, \\
 & \phi' = \frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
 \mathbf{TS}: \quad & \sigma' = -\frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi, \\
 & \phi' = -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
 \mathbf{ST}: \quad & \sigma' = -\frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi, \\
 & \phi' = +\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
 \mathbf{TST}: \quad & \sigma' = \frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi, \\
 & \phi' = -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi.
 \end{aligned} \tag{28}$$

Instead of realizing the discrete symmetries on the two scalars σ and ϕ one may alternatively describe them by their action on the compactification radius R and the string coupling constant g_s which are related to σ and ϕ via

$$R = e^{-2\sigma/\sqrt{3}}, \quad g_s = e^{\phi - \sigma/\sqrt{3}}. \tag{29}$$

In terms of R and g_s the D_3 -rules are given by¹²

$$\begin{aligned}
 \mathbf{e}: \quad & R' = R, \quad g'_s = g_s, \\
 \mathbf{T}: \quad & R' = \frac{1}{R}, \quad g'_s = \frac{g_s}{R}, \\
 \mathbf{S}: \quad & R' = \frac{R}{g_s}, \quad g'_s = \frac{1}{g_s}, \\
 \mathbf{TS}: \quad & R' = \frac{1}{g_s}, \quad g'_s = \frac{R}{g_s}, \\
 \mathbf{ST}: \quad & R' = \frac{g_s}{R}, \quad g'_s = \frac{1}{R},
 \end{aligned}$$

¹¹ For future reference we have given each element a name. Note that the **T**-element corresponds to the usual T -duality and that the **S**-element corresponds to the string/string duality. Furthermore, by **TS** we mean the symmetry that is obtained by a composition of **T** and **S** as follows:

$$\sigma'' = \frac{1}{2}\sigma' + \frac{1}{2}\sqrt{3}\phi' = -\frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi. \tag{27}$$

¹² Some of these rules have already been given in [7]. Note that the **T**-element inverts the compactification radius R while the **S**-element inverts the string coupling constant g_s , as expected.

Table 2

Group multiplication table of the 6-element dihedral group D_3

	e	T	S	ST	TS	TST
e	e	T	S	ST	TS	TST
T	T	e	TS	TST	S	ST
S	S	ST	e	T	TST	TS
ST	ST	TST	T	TS	e	T
TS	TS	S	TST	e	ST	S
TST	TST	TS	ST	S	T	e

$$\mathbf{TST} : \quad R' = g_s, \quad g'_s = R. \quad (30)$$

To every element of D_3 corresponds four elements of C/\mathbb{Z}_2 acting on the three vectors. The specific transformations of the vectors are given by¹³

$$\begin{aligned}
 \mathbf{e} : \quad & A' = A, \quad B' = B, \quad C' = C, \\
 \mathbf{T} : \quad & A' = B, \quad B' = A, \quad C' = C, \\
 \mathbf{S} : \quad & A' = A, \quad B' = C, \quad C' = B, \\
 \mathbf{TS} : \quad & A' = B, \quad B' = C, \quad C' = A, \\
 \mathbf{ST} : \quad & A' = C, \quad B' = A, \quad C' = B, \\
 \mathbf{TST} : \quad & A' = C, \quad B' = B, \quad C' = A.
 \end{aligned} \quad (31)$$

Finally, for the convenience of the reader we collect below some useful properties of D_3 . The dihedral group D_3 is defined by the property that it has a 2-order element a and a 3-order element b such that ab is 2-order, i.e.

$$a^2 = b^3 = (ab)^2 = e, \quad (32)$$

where e is the unit element. We may take $a = \mathbf{T}$ and $b = \mathbf{ST}$. Note that \mathbf{e} is 1-order, \mathbf{S} , \mathbf{T} , \mathbf{TST} are 2-order and \mathbf{ST} , \mathbf{TS} are 3-order. Some useful relations are

$$(\mathbf{ST})^2 = \mathbf{TS}, \quad (\mathbf{TS})^2 = \mathbf{ST}, \quad \mathbf{STS} = \mathbf{TST}. \quad (33)$$

The complete group multiplication table of D_3 is given in Table 2.

This concludes our discussion of the discrete duality symmetries that act in the common sector. Clearly, any of the symmetries given above can be formulated as a 6-dimensional duality symmetry by using the inverse of the reduction formulae given in Eq. (17). We next discuss the type II dualities. For that purpose we will first introduce in the next section the different theories involved. The type II dualities¹⁴ that act within and between these theories will be discussed in Section 5.

¹³ We only give six elements of C/\mathbb{Z}_2 . To every element below one can associate three more elements by changing (in three possible ways) *two* signs in the given transformation rules.

¹⁴ In $D = 10$ the name type I, type II is related to $N = 1, N = 2$ supersymmetry, respectively. In $D = 6$ we have $N = 1, 2$ and $N = 4$ supersymmetry. Since the theories we will discuss have $N = 2$ supersymmetry we denote their duality symmetries as type II dualities.

3. Heterotic, type IIA and type IIB in six dimensions

In this section we will describe and discuss the action and non-compact symmetries of (i) the 10-dimensional heterotic string compactified on T^4 , (ii) the 10-dimensional type IIA string compactified on $K3$ and (iii) the 10-dimensional type IIB string compactified on $K3$. The dimensional reduction of these theories from $D = 6$ to $D = 5$ will be discussed in the next section.

3.1. Heterotic

The field content of the heterotic theory is given by the usual metric, dilaton, anti-symmetric tensor system $\{\hat{g}, \hat{\phi}, \hat{B}\}$ plus 80 scalars and 24 Abelian gauge fields. The 80 scalars parametrize an $O(4, 20)/(O(4) \times O(20))$ coset and are combined into the symmetric 24×24 -dimensional matrix \hat{M}^{-1} satisfying $\hat{M}^{-1} L \hat{M}^{-1} = L$ where L is the invariant metric on $O(4, 20)$:

$$L = \begin{Bmatrix} 0 & \mathbb{I}_4 & 0 \\ \mathbb{I}_4 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_{16} \end{Bmatrix}. \quad (34)$$

The heterotic action in the string-frame metric is given by

$$I_H = \frac{1}{2} \int d^6 x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left[-\hat{R} + 4|d\hat{\phi}|^2 - \frac{3}{4}|\hat{H}|^2 + \frac{1}{8} \text{Tr} \left(\partial_{\hat{\mu}} \hat{M} \partial^{\hat{\mu}} \hat{M}^{-1} \right) - (d\hat{V})_{\hat{\mu}\hat{\nu}}^a \hat{M}_{ab}^{-1} (d\hat{V})^{\hat{\mu}\hat{\nu}b} \right], \quad (35)$$

where \hat{H} is defined by

$$\hat{H} = d\hat{B} + \hat{V}^a d\hat{V}^b L_{ab}. \quad (36)$$

The Chern–Simons term inside \hat{H} is related to the following transformation rule of \hat{B} under the Abelian gauge transformations with parameter $\hat{\eta}$:

$$\delta \hat{B} = -\hat{V}^a d\hat{\eta}^b L_{ab}. \quad (37)$$

The heterotic action (35) is invariant under a non-compact $O(4, 20)$ symmetry

$$\hat{V}'_{\hat{\mu}} = \Omega \hat{V}_{\hat{\mu}}, \quad (\hat{M}^{-1})' = \Omega \hat{M}^{-1} \Omega^T, \quad (38)$$

where Ω is an element of $O(4, 20)$.

3.2. Type IIA

The field content of the type IIA theory in six dimensions is identical to the heterotic theory. The action, however, is different. Instead of a Chern–Simons term inside \hat{H} the

action contains an additional topological term as compared to the heterotic action. We thus have¹⁵

$$\begin{aligned}
 I_{\text{IIA}} = & \frac{1}{2} \int_{\mathcal{M}_6} d^6 x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left[-\hat{R} + 4|d\hat{\phi}|^2 - \frac{3}{4}|d\hat{B}|^2 \right. \\
 & \left. + \frac{1}{8} \text{Tr} \left(\partial_{\hat{\mu}} \hat{M} \partial^{\hat{\mu}} \hat{M}^{-1} \right) - e^{2\hat{\phi}} (d\hat{V})_{\hat{\mu}\hat{\nu}}^a \hat{M}_{ab}^{-1} (d\hat{V})^{\hat{\mu}\hat{\nu}b} \right] \\
 & - \frac{1}{8} \int_{\mathcal{M}_7} d\hat{B} d\hat{V}^a d\hat{V}^b L_{ab}, \quad (39)
 \end{aligned}$$

where \mathcal{M}_7 is a 7-manifold with boundary \mathcal{M}_6 .

3.3. Type IIB

The field content of the type IIB theory is given by a metric, five self-dual antisymmetric tensors, 21 anti-self-dual antisymmetric tensors and 105 scalars. The 105 scalars parametrize an $O(5, 21)/(O(5) \times O(21))$ coset and are combined into the symmetric 26×26 -dimensional matrix $\hat{\mathcal{M}}$ satisfying the condition $\hat{\mathcal{M}}^{-1} \mathcal{L} \hat{\mathcal{M}}^{-1} = \mathcal{L}$ where \mathcal{L} is the invariant metric on $O(5, 21)$:

$$\mathcal{L} = \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & L \end{Bmatrix}, \quad (40)$$

and L is the invariant metric on $O(4, 20)$ given in (34). The type IIB theory is obtained by a reduction over $K3$ of the 10-dimensional type IIB theory [16]. Its field equations were constructed in [17].

A complicating feature of the type IIB theory is that, due to the self-duality conditions on the antisymmetric tensors, there is no Lorentz-covariant action [18]. However, there exists a so-called non-self-dual action that has the property that its field equations lead to the correct type IIB field equations upon substituting the self-duality constraints by hand. A similar action has been introduced for the $D = 10$ type IIB theory [19]. The non-self-dual action also occurs naturally in the group-manifold approach, both in $D = 6$ [20] as well as $D = 10$ [21]¹⁶. An important property of the non-self-dual action is that it leads to the correct action for the type IIB theory *after* dimensional reduction. Therefore, the non-self-dual type IIB action is well suited for our purposes. The reason for this is that the self-duality conditions after the dimensional reduction become algebraic conditions that can be substituted back into the effective action. We will show how this works in

¹⁵ Although we are dealing with a different theory we will indicate the type IIA fields with the same symbols as the heterotic fields. Whenever confusion could arise, we will explicitly denote whether a field is to be considered as a heterotic or a type IIA field.

¹⁶ In the group manifold approach the self-duality constraints follow from varying a non-space-time field. Put differently, only those space-time background fields are allowed for a description using the group manifold approach that satisfy the self-duality constraints.

more detail in the next section. We find that in the Einstein-frame the non-self-dual type IIB action is given by

$$I_{\text{IIB}} = \frac{1}{2} \int d^6 x \sqrt{-\hat{g}} \left[-\hat{R} + \frac{1}{8} \text{Tr}(\partial_{\hat{\mu}} \hat{\mathcal{M}} \partial^{\hat{\mu}} \hat{\mathcal{M}}^{-1}) + \frac{3}{8} (d\hat{B})^i_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\mathcal{M}}^{-1}_{ij} (d\hat{B})^{\hat{\mu}\hat{\nu}\hat{\rho}j} \right]. \quad (41)$$

The field equations corresponding to this action lead to the correct type IIB field equations provided we substitute by hand the following (anti-) self-duality conditions for the antisymmetric tensors \hat{B}^i ($i = 1, \dots, 26$):

$$(d\hat{B})^i = \mathcal{L}^{ij} \hat{\mathcal{M}}^{-1}_{jk} * (d\hat{B})^k. \quad (42)$$

In order to extract the common sector out of the type IIB theory, it is necessary to use a particular parametrization of the matrix $\hat{\mathcal{M}}^{-1}$ in terms of the 105 scalars, thereby identifying a particular scalar as the type IIB dilaton $\hat{\phi}$. This dilaton may then be used to define a string-frame metric \hat{g}_S via $\hat{g}_S = e^{\hat{\phi}} \hat{g}_E$ where \hat{g}_E is the Einstein-frame metric. We use the following parametrization:

$$\hat{\mathcal{M}}^{-1} = \begin{pmatrix} -e^{-2\hat{\phi}} + \hat{\ell}^a \hat{\ell}^b \hat{M}_{ab}^{-1} - \frac{1}{4} e^{2\hat{\phi}} \hat{\ell}^4 & \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 & \hat{\ell}^a \hat{M}_{ab}^{-1} - \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 \hat{\ell}^a L_{ab} \\ \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 & -e^{2\hat{\phi}} & e^{2\hat{\phi}} \hat{\ell}^a L_{ab} \\ \hat{\ell}^a \hat{M}_{ab}^{-1} - \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 \hat{\ell}^a L_{ab} & e^{2\hat{\phi}} \hat{\ell}^a L_{ab} & \hat{M}_{ab}^{-1} - e^{2\hat{\phi}} \hat{\ell}^c \hat{\ell}^d L_{ac} L_{bd} \end{pmatrix}, \quad (43)$$

where 80 scalars are contained in the $O(4, 20)$ matrix \hat{M}^{-1} , 24 scalars are described by the $\hat{\ell}^a$ and where $\hat{\phi}$ is identified as the type IIB dilaton. Furthermore, we have used the definition

$$\hat{\ell}^2 \equiv \hat{\ell}^a \hat{\ell}^b L_{ab} \quad (a = 1, \dots, 24). \quad (44)$$

The common sector is obtained by imposing the constraints:

$$\hat{B}^i_{\hat{\mu}\hat{\nu}} = 0 \quad (i = 3, \dots, 26), \quad \hat{\ell}^a = 0, \quad \hat{M}_{ab}^{-1} = \delta_{ab}. \quad (45)$$

After imposing these constraints the (anti-) self-duality conditions (42) reduce to

$$d\hat{B}^{(2)} = -e^{-2\hat{\phi}} * d\hat{B}^{(1)}. \quad (46)$$

Substituting the constraints (45) and the constraint (46) back into the type IIB action (41) one obtains the standard form of the action for the common sector in the Einstein metric as given in (14). Having identified the type IIB dilaton it is straightforward to convert this result to the string-frame metric as given in (12). Note that although the type IIB theory has no Lorentz-covariant action, the common subsector does. This is due to the fact that the corresponding (anti-) self-duality constraint (46) is off-diagonal and hence can be used to eliminate e.g. $\hat{B}^{(2)}$ in terms of $\hat{B}^{(1)}$.

4. Dimensional reduction

In this section we describe the dimensional reduction to $D = 5$ of the $D = 6$ theories described in the previous section. As a main result we will show that by using particular

dimensional reduction schemes in each case the $D = 6$ heterotic, type IIA and type IIB theories can be mapped onto the *same* $D = 5$ type II theory. For reasons, explained in the introduction, we will associate each of the reduction formulae with an element of D_3 . The names of the reduction formulae (and their inverse) are chosen such that, when restricted to the common subsector, each reduction formula can be obtained from that of the heterotic string (which we have chosen as the unit element) by the action of the corresponding element of D_3 . In the next section we will use the reduction formulae, derived in this section, as building blocks to construct the different discrete duality symmetries acting in six dimensions.

4.1. Heterotic

We make the following ansatz for the 6-dimensional fields in terms of the 5-dimensional fields¹⁷:

$$e : \begin{cases} \hat{g}_{xx} = -e^{-4\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} = -e^{-4\sigma/\sqrt{3}} A_\mu, \\ \hat{g}_{\mu\nu} = g_{\mu\nu} - e^{-4\sigma/\sqrt{3}} A_\mu A_\nu, \\ \hat{B}_{\mu\nu} = B_{\mu\nu}^{(C)} + A_{[\mu} B_{\nu]} + \ell^a V_{[\mu}^b A_{\nu]} L_{ab}, \\ \hat{B}_{x\mu} = B_\mu - \frac{1}{2} \ell^a V_\mu^b L_{ab}, \\ \hat{\phi} = \phi - \frac{1}{\sqrt{3}} \sigma, \\ \hat{V}_\mu^a = V_\mu^a + \ell^a A_\mu, \\ \hat{\ell}^a = \ell^a, \\ \hat{M} = M. \end{cases} \quad (47)$$

The Latin superscript of the 5-dimensional antisymmetric tensor indicates to which vector this tensor needs to be dualized in order to actually obtain the common $D = 5$ type II theory. This dualization goes via the formula

$$H^{\mu\nu\rho(C)} = \frac{1}{3\sqrt{g}} e^{2\phi} \epsilon^{\mu\nu\rho\lambda\sigma} (dC)_{\lambda\sigma}. \quad (48)$$

The inverse relations corresponding to (47) are given by

$$e^{-1} : \begin{cases} g_{\mu\nu} = \hat{g}_{\mu\nu} - \hat{g}_{x\mu} \hat{g}_{x\nu} / \hat{g}_{xx}, \\ B_{\mu\nu}^{(C)} = \hat{B}_{\mu\nu} + \hat{g}_{x[\mu} \hat{B}_{\nu]} / \hat{g}_{xx} + \frac{1}{2} \hat{V}_x^a \hat{g}_{x[\mu} \hat{V}_{\nu]}^b L_{ab} / \hat{g}_{xx}, \\ \phi = \hat{\phi} - \frac{1}{4} \log(-\hat{g}_{xx}), \\ A_\mu = \hat{g}_{x\mu} / \hat{g}_{xx}, \\ B_\mu = \hat{B}_{x\mu} + \frac{1}{2} \hat{V}_x^a \hat{V}_\mu^b L_{ab} - \frac{1}{2} \hat{V}_x^a \hat{V}_x^b L_{ab} \hat{g}_{x\mu} / \hat{g}_{xx}, \\ V_\mu^a = \hat{V}_\mu^a - \hat{V}_x^a \hat{g}_{x\mu} / \hat{g}_{xx}, \\ \sigma = -\frac{1}{4} \sqrt{3} \log(-\hat{g}_{xx}), \\ \ell^a = \hat{V}_x^a, \\ M = \hat{M}. \end{cases} \quad (49)$$

¹⁷ In this section we will always use the string-frame metric both in $D = 6$ as well as in $D = 5$.

The dimensionally reduced action in the (5-dimensional) string-frame metric is given by

$$I_{II} = \frac{1}{2} \int_{\mathcal{M}_5} d^5x \sqrt{g} e^{-2\phi} \left\{ -R + 4|d\phi|^2 + \frac{1}{8} \text{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) \right. \\ \left. + e^{4\phi} |dC|^2 - (d\mathcal{A})_{\mu\nu}^i \mathcal{M}_{ij}^{-1} (d\mathcal{A})^{\mu\nu j} \right\} - \frac{1}{4} \int_{\mathcal{M}_6} dC d\mathcal{A}^i d\mathcal{A}^j \mathcal{L}_{ij}, \quad (50)$$

where \mathcal{M}_6 is a 6-manifold with boundary \mathcal{M}_5 and \mathcal{L} is the invariant metric on $O(5, 21)$ given in (40). The vectors \mathcal{A}^i ($i = 1, \dots, 26$) are given by

$$\mathcal{A}_\mu^i = \begin{Bmatrix} A_\mu \\ B_\mu \\ V_\mu^a \end{Bmatrix}. \quad (51)$$

The explicit expression of the $O(5, 21)$ matrix \mathcal{M} in terms of the 105 scalars σ, ℓ^a and the 80 scalars contained in the $O(4, 20)$ matrix M is given by

$$\mathcal{M}^{-1} = \begin{Bmatrix} -e^{-4\sigma/\sqrt{3}} + \ell^a \ell^b M_{ab}^{-1} - \frac{1}{4} e^{4\sigma/\sqrt{3}} \ell^4 & \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 & \ell^a M_{ab}^{-1} - \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 \ell^a L_{ab} \\ \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 & -e^{4\sigma/\sqrt{3}} & e^{4\sigma/\sqrt{3}} \ell^a L_{ab} \\ \ell^a M_{ab}^{-1} - \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 \ell^a L_{ab} & e^{4\sigma/\sqrt{3}} \ell^a L_{ab} & M_{ab}^{-1} - e^{4\sigma/\sqrt{3}} \ell^c \ell^d L_{ac} L_{bd} \end{Bmatrix}, \quad (52)$$

where $\ell^2 \equiv \ell^a \ell^b L_{ab}$. These scalars parametrize the coset $O(5, 21)/(O(5) \times O(21))$.

The action (50) defines the type II theory in five dimensions¹⁸. It clearly contains the common sector given in (21). This may be seen by imposing the following constraints:

$$\ell^a = V_\mu^a = 0, \quad M_{ab}^{-1} = \delta_{ab}. \quad (53)$$

Finally, the $D = 5$ type II action in the (5-dimensional) Einstein-frame metric takes the following form:

$$I_{II} = \frac{1}{2} \int_{\mathcal{M}_5} d^5x \sqrt{g} \left\{ -R - \frac{4}{3} |d\phi|^2 + \frac{1}{8} \text{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) \right. \\ \left. + e^{8\phi/3} |dC|^2 - e^{-4\phi/3} (d\mathcal{A})_{\mu\nu}^i \mathcal{M}_{ij}^{-1} (d\mathcal{A})^{\mu\nu j} \right\} - \frac{1}{4} \int_{\mathcal{M}_6} dC d\mathcal{A}^i d\mathcal{A}^j \mathcal{L}_{ij}. \quad (54)$$

4.2. Type IIA

Clearly, the dimensional reduction of the type IIA theory leads to a type II theory in five dimensions similar to the one given above. We also know that the type II theory in five dimensions is *unique*. Therefore, it should be possible to map the $D = 6$ type

¹⁸ The string dynamics corresponding to this $D = 5$ type II theory has been analyzed recently in Refs. [22,23].

IIA theory onto the *same* $D = 5$ type II theory that we obtained above by dimensional reduction of the heterotic theory. To achieve this, one must use a particular reduction scheme of the type IIA theory that is *different* from that of the heterotic theory described above.

To be precise, we find that the dimensional reduction of the $D = 6$ type IIA theory leads to exactly the *same* $D = 5$ type II theory defined in Eq. (50) provided we use the following dimensional reduction formulae for the type IIA theory:

$$S : \begin{cases} \hat{g}_{\underline{x}\underline{x}} = -e^{-2\phi-2\sigma/\sqrt{3}}, \\ \hat{g}_{\underline{x}\mu} = -e^{-2\phi-2\sigma/\sqrt{3}} A_\mu, \\ \hat{g}_{\mu\nu} = e^{-2\phi+2\sigma/\sqrt{3}} g_{\mu\nu} - e^{-2\phi-2\sigma/\sqrt{3}} A_\mu A_\nu, \\ \hat{B}_{\mu\nu} = B_{\mu\nu}^{(B)} + A_{[\mu} C_{\nu]}, \\ \hat{B}_{\underline{x}\mu} = C_\mu, \\ \hat{\phi} = -\phi + \frac{1}{\sqrt{3}}\sigma, \\ \hat{V}_\mu^a = V_\mu^a + \ell^a A_\mu, \\ \hat{V}_{\underline{x}}^a = \ell^a, \\ \hat{M} = M. \end{cases} \quad (55)$$

The 5-dimensional antisymmetric tensor $B_{\mu\nu}^{(B)}$ is dualized to a vector B_μ via the relation

$$H^{\mu\nu\rho(B)} = \frac{1}{3\sqrt{g}} e^{2\phi+4\sigma/\sqrt{3}} \epsilon^{\mu\nu\rho\lambda\sigma} \left[(dB)_{\lambda\sigma} + \ell^a (dV)_{\lambda\sigma}^b L_{ab} + \ell^2 (dA)_{\lambda\sigma} \right]. \quad (56)$$

The inverse relations corresponding to (55) are given by

$$S^{-1} : \begin{cases} g_{\mu\nu} = e^{-2\hat{\phi}} \left(\hat{g}_{\mu\nu} - \hat{g}_{\underline{x}\mu} \hat{g}_{\underline{x}\nu} / \hat{g}_{\underline{x}\underline{x}} \right), \\ B_{\mu\nu}^{(B)} = \hat{B}_{\mu\nu} + \hat{g}_{\underline{x}[\mu} \hat{B}_{\nu]\underline{x}} / \hat{g}_{\underline{x}\underline{x}}, \\ \phi = -\frac{1}{2} \hat{\phi} - \frac{1}{4} \log(-\hat{g}_{\underline{x}\underline{x}}), \\ A_\mu = \hat{g}_{\underline{x}\mu} / \hat{g}_{\underline{x}\underline{x}}, \\ C_\mu = \hat{B}_{\underline{x}\mu}, \\ V_\mu^a = \hat{V}_\mu^a - \hat{V}_{\underline{x}}^a \hat{g}_{\underline{x}\mu} / \hat{g}_{\underline{x}\underline{x}}, \\ \sigma = \frac{\sqrt{3}}{2} \hat{\phi} - \frac{\sqrt{3}}{4} \log(-\hat{g}_{\underline{x}\underline{x}}), \\ \ell^a = \hat{V}_{\underline{x}}^a, \\ M = \hat{M}. \end{cases} \quad (57)$$

4.3. Type IIB

The above discussion for the type IIA theory also applies to the type IIB theory. We find that the dimensional reduction of the type IIB theory leads to the *same* $D = 5$ type II theory (50) as the dimensional reduction of the heterotic and type IIA theory provided we use the following dimensional reduction formulae for the type IIB fields:

$$\text{ST : } \begin{cases} \hat{g}_{xx} = -e^{2\phi+2\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} = -e^{2\phi+2\sigma/\sqrt{3}} C_\mu, \\ \hat{g}_{\mu\nu} = e^{-2\phi+2\sigma/\sqrt{3}} g_{\mu\nu} - e^{2\phi+2\sigma/\sqrt{3}} C_\mu C_\nu, \\ \hat{B}_{\mu\nu}^i = B_{\mu\nu}^{(\mathcal{A})i} + C_{[\mu} \mathcal{A}_{\nu]}^i, \\ \hat{B}_{x\mu}^i = \mathcal{A}_\mu^i, \\ \hat{\phi} = \frac{2}{\sqrt{3}} \sigma, \\ \hat{\ell}^a = \ell^a, \\ \hat{M} = M. \end{cases} \quad (58)$$

Note that due to the (anti-) self-duality relations (42) both $\hat{B}_{\mu\nu}^i$ as well as $\hat{B}_{x\mu}^i$ get related to the 5-dimensional vector fields \mathcal{A}_μ^i . This goes as follows. In a first stage the dimensional reduction of $\hat{B}_{\mu\nu}^i$ leads to a 5-dimensional antisymmetric tensor $B_{\mu\nu}^{(\mathcal{B})i}$ that may be dualized to a 5-dimensional vector \mathcal{B}_μ^i . To perform this dualization one adds an extra term to the kinetic term of $B_{\mu\nu}^{(\mathcal{B})i}$ containing \mathcal{B}_μ^i as a Lagrange multiplier for the Bianchi identity of $B_{\mu\nu}^{(\mathcal{B})i}$. The two terms together are given by

$$\frac{3}{16} \sqrt{g} e^{2\phi} H_{\mu\nu\rho}^i \mathcal{M}_{ij}^{-1} H^{j\mu\nu\rho} + \frac{1}{8} \epsilon^{\mu\nu\rho\lambda\sigma} \mathcal{B}_\mu^i \left(\partial_\nu H_{\rho\lambda\sigma}^j - 2\partial_\nu C_\rho \partial_\lambda \mathcal{A}_\sigma^j \right) \mathcal{L}_{ij}, \quad (59)$$

where H^i is now considered as an *independent* field. The field equation of $H_{\mu\nu\rho}^i$ leads to the identity

$$H^{k\rho\sigma\lambda} = -\frac{1}{3\sqrt{g}} e^{-2\phi} \epsilon^{\mu\nu\rho\sigma\lambda} \mathcal{M}^{ki} (d\mathcal{B})_{\mu\nu}^j \mathcal{L}_{ij}. \quad (60)$$

At the same time, the dimensional reduction of the $D = 6$ (anti-) self-duality constraint (42) leads to exactly the same identity in $D = 5$ except that \mathcal{B}_μ^i is replaced by \mathcal{A}_μ^i :

$$H^{k\rho\sigma\lambda} = -\frac{1}{3\sqrt{g}} e^{-2\phi} \epsilon^{\mu\nu\rho\sigma\lambda} \mathcal{M}^{ki} (d\mathcal{A})_{\mu\nu}^j \mathcal{L}_{ij}. \quad (61)$$

Combining these two results leads to the conclusion that the original $D = 6$ constraint (42) has become a simple algebraic relation in $D = 5$:

$$\mathcal{B}_\mu^i = \mathcal{A}_\mu^i. \quad (62)$$

Substituting the field equation of $H_{\mu\nu\rho}^i$, Eq. (60), with the above identification understood, back into (59) then leads to the desired result. Note that the last term in (59) leads to the topological term which is present in the 5-dimensional type II action (50).

Finally, the inverse relations are given by

$$(\text{ST})^{-1} : \left\{ \begin{array}{l} g_{\mu\nu} = -e^{-2\hat{\phi}} \hat{g}_{\underline{x}\underline{x}} (\hat{g}_{\mu\nu} - \hat{g}_{\underline{x}\mu} \hat{g}_{\underline{x}\nu} / \hat{g}_{\underline{x}\underline{x}}) , \\ B_{\mu\nu}^{(\mathcal{A})i} = \hat{B}_{\mu\nu}^i + \hat{g}_{\underline{x}[\mu} \hat{B}_{\nu]\underline{x}}^i / \hat{g}_{\underline{x}\underline{x}} , \\ \phi = -\frac{1}{2} \hat{\phi} + \frac{1}{2} \log(-\hat{g}_{\underline{x}\underline{x}}) , \\ \mathcal{A}_{\mu}^i = \hat{B}_{\underline{x}\mu}^i , \\ \sigma = \frac{1}{2} \sqrt{3} \hat{\phi} , \\ \ell^a = \hat{\ell}^a , \\ M = \hat{M} . \end{array} \right. \quad (63)$$

5. Type II dualities

In this section we will show how the reduction formulae constructed in the previous section may be used in a systematic way to construct discrete duality symmetries in six dimensions that act within and between the heterotic, type IIA and type IIB theories.

We first note that to each reduction formula given in the previous section one can associate three further reduction formulae. This is due to the fact that although the $D = 5$ type II theory given in (50) is not invariant under the full 24-element proper cubic group it is still invariant under a 4-element $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, i.e.

$$\mathcal{C}/\mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 . \quad (64)$$

This formula is the 5-dimensional analogue of the $D = 9$ formula given in (9). In particular the T -duality symmetry remains unbroken¹⁹. Its explicit form in $D = 5$ is given by the following particular \mathbb{Z}_2 subgroup of the non-compact $O(5, 21)$ symmetry with parameter Ω given by

$$\Omega = \mathcal{L} , \quad (65)$$

where \mathcal{L} is the flat metric given in Eq. (40). In terms of components the $D = 5$ T -duality rules in the string-frame metric read

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu} , \\ \phi' &= \phi , \\ e^{-4\sigma'/\sqrt{3}} &= e^{-4\sigma/\sqrt{3}} \left(e^{-8\sigma/\sqrt{3}} - e^{-4\sigma/\sqrt{3}} \ell^a \ell^b M_{ab}^{-1} + \frac{1}{4} \ell^4 \right)^{-1} , \\ \ell'^a &= \frac{e^{-4\sigma/\sqrt{3}} \ell^c M_{cd}^{-1} L^{da} - \frac{1}{2} \ell^2 \ell^a}{e^{-8\sigma/\sqrt{3}} - e^{-4\sigma/\sqrt{3}} \ell^c \ell^d M_{cd}^{-1} + \frac{1}{4} \ell^4} , \\ M' &= M^{-1} , \\ A'_{\mu} &= B_{\mu} , \\ B'_{\mu} &= A_{\mu} , \\ C'_{\mu} &= C_{\mu} , \end{aligned}$$

¹⁹ Note that this is *different* from the situation in nine dimensions where the T -duality symmetry is broken.

$$V'_\mu = LV_\mu. \quad (66)$$

Note that, when restricted to the common sector these rules reduce to the standard ones given in Section 2. We also observe that, for the special case that $M_{ab} = \delta_{ab}$, the complicated duality rules of σ and ℓ^a factorize such that the final result can be written in terms of an effective metric as was discussed in [12]. This effective metric has a natural origin in a sigma model approach [24]. Apparently this factorization does not occur for non-trivial values of the matrix M .

The second \mathbb{Z}_2 -factor in (64) corresponds to another \mathbb{Z}_2 subgroup of $O(5, 21)$ with parameter Ω given by

$$\Omega = -\mathbb{I}. \quad (67)$$

Its action in components is given by

$$\mathcal{A}_\mu^{i\prime} = -\mathcal{A}_\mu^i, \quad (68)$$

while all other fields remain invariant. To simplify the discussion below we will from now on only concentrate on those transformations that act non-trivially on the scalars. In that case we are only dealing with the dihedral group D_3 which gets broken to

$$D_3 \rightarrow \mathbb{Z}_2, \quad (69)$$

where the \mathbb{Z}_2 factor is given by the T -duality transformation above. The presence of this T -duality symmetry in five dimensions means that to each reduction formula we can associate a so-called T -dual version. Its explicit form is obtained by replacing in the original reduction formula each 5-dimensional fields by its T -dual expression. The T -dual reduction formula so obtained should lead to the same answer in five dimensions. This is guaranteed by the fact that the 5-dimensional action is invariant under T -duality. We will indicate the T -dual versions of the reduction formulae constructed in the previous section as follows:

$$\begin{aligned} \mathbf{e} &\rightarrow \mathbf{T}, \\ \mathbf{S} &\rightarrow \mathbf{TS}, \\ \mathbf{ST} &\rightarrow \mathbf{TST}. \end{aligned} \quad (70)$$

We thus obtain six different reduction formulae which correspond to the three down-pointing arrows in Fig. 6. Similarly, there are six inverse reduction, or so-called decompactification, formulae which go opposite the vertical arrows in Fig. 6. These decompactification formulae will be indicated by the inverse group elements. The claim is now that, using these 6 reduction and decompactification formulae only, one is able to construct in a simple way all the discrete dualities that act within and between the heterotic, type IIA and type IIB theories that are indicated in Fig. 6. Each discrete duality symmetry has been given a name such that, when restricted to the common subsector, the duality becomes the corresponding D_3 duality symmetry that acts in the common subsector.

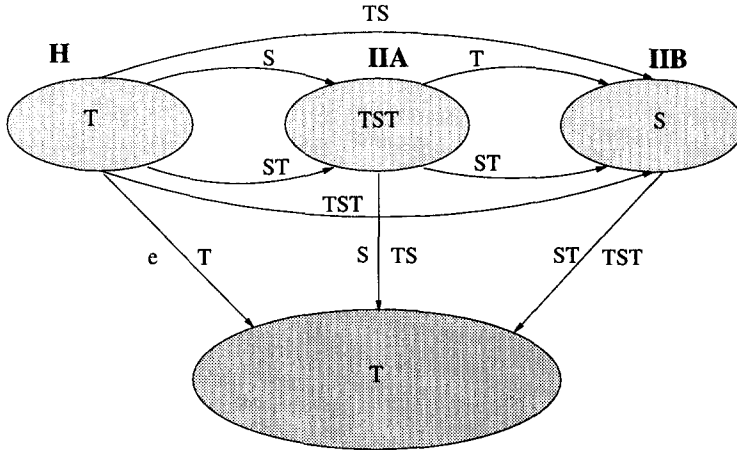


Fig. 6. The three down-pointing arrows indicate the six possible ways to map the three $D = 6$ theories (heterotic, type IIA, type IIB) onto the *same* $D = 5$ type II theory. Each reduction formula is indicated by a (boldface) element of D_3 . As explained in the text these six reduction formula and their inverses may be used to construct the explicit form of all the discrete $D = 6$ dualities that are indicated in the figure.

To show how the dualities of Fig. 6 may be constructed starting from the different reduction and decompactification formulae it is instructive to first give a few examples.

- (1) The T -duality that acts within the heterotic theory is obtained by first reducing the heterotic theory using the \mathbf{e} reduction formulae given in (47) and next the decompactification formula \mathbf{T}^{-1} defined in (70), i.e.

$$T(H \rightarrow H) = \mathbf{T}^{-1} \times \mathbf{e} = \mathbf{T}. \quad (71)$$

- (2) The S -duality that maps the heterotic onto the type IIA theory is obtained by first reducing the heterotic theory with \mathbf{e} and next oxidizing the $D = 5$ theory with \mathbf{S}^{-1} . As Fig. 6 shows there are three other possibilities, one of them gives the same answer while the other two are related to the ST map indicated in Fig. 6:

$$\begin{aligned} S(H \rightarrow \text{IIA}) &= \mathbf{S}^{-1} \times \mathbf{e} = \mathbf{S}^{-1} = \mathbf{S}, \\ S(H \rightarrow \text{IIA}) &= (\mathbf{TS})^{-1} \times \mathbf{T} = \mathbf{ST} \times \mathbf{T} = \mathbf{S}, \\ (ST)(H \rightarrow \text{IIA}) &= \mathbf{S}^{-1} \times \mathbf{T} = \mathbf{S} \times \mathbf{T} = \mathbf{ST}, \\ (ST)(H \rightarrow \text{IIA}) &= (\mathbf{TS})^{-1} \times \mathbf{e} = \mathbf{ST} \times \mathbf{e} = \mathbf{ST}. \end{aligned} \quad (72)$$

Note that we have used here the group multiplication table of D_3 given in Table 2. The reason that we are allowed to use the group multiplication table of D_3 is that we have set up our notation for the six reduction and decompactification formulae in such a way that, when restricted to the common subsector, these reduction and decompactification formulae actually become specific D_3 symmetries in five dimensions.

- (3) The S -duality that acts within the IIB theory is obtained by first reducing the IIB theory with \mathbf{ST} and then oxidizing with $(\mathbf{TST})^{-1}$. The other way round gives the

same answer:

$$\begin{aligned}
 S(\text{IIB} \rightarrow \text{IIB}) &= (\text{TST})^{-1} \times \text{ST} = \text{TST} \times \text{ST} \\
 &= \text{STS} \times \text{ST} = \text{S}, \\
 S(\text{IIB} \rightarrow \text{IIB}) &= (\text{ST})^{-1} \times \text{TST} = \text{TS} \times \text{TST} \\
 &= \text{TS} \times \text{STS} = \text{S},
 \end{aligned} \tag{73}$$

where we have used some of the D_3 identities given in Eq. (33).

- (4) We deduce from Fig. 6 that there is not only a T -duality that acts within the heterotic theory but also a T -duality that maps the IIA theory onto the IIB theory. It may be obtained in the following two ways from the reduction/decompactification formulae:

$$\begin{aligned}
 T(\text{IIA} \rightarrow \text{IIB}) &= (\text{ST})^{-1} \times \text{S} = \text{TS} \times \text{S} = \text{T}, \\
 T(\text{IIA} \rightarrow \text{IIB}) &= (\text{TST})^{-1} \times \text{TS} = \text{TST} \times \text{TS} = \text{T}.
 \end{aligned} \tag{74}$$

- (5) Finally, we observe that ST is a 3-order element of D_3 . This means that starting with the heterotic theory and applying the ST -duality three times we should get back the heterotic theory. In the diagram of Fig. 6 this is seen as follows: The first ST duality brings us to the IIA theory, the second one brings us from the IIA to the IIB theory. Finally, to perform the last ST duality we observe that $ST = (TS)^{-1}$, i.e. this duality brings us back from the type IIB theory to the heterotic theory via the opposite direction of the oriented arrow at the top of the diagram.

The above examples should explain the main idea of how the reduction and decompactification formulae of the previous section are used to construct the different discrete duality symmetries in six dimensions. As a further illustration of our method we will now give the explicit expression of three of such $D = 6$ dualities.

- (A) Clearly, the S -duality map from the heterotic onto the type IIA theory should reproduce the known $D = 6$ string/string duality rule [5]. It indeed does and we find that the S duality is given by (using the string-frame metric):

$$\begin{aligned}
 \hat{G}_{\hat{\mu}\hat{\nu}} &= e^{-2\hat{\phi}} \hat{g}_{\hat{\mu}\hat{\nu}}, \\
 \hat{\Phi} &= -\hat{\phi}, \\
 \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= e^{-2\hat{\phi}} * \hat{h}_{\hat{\mu}\hat{\nu}\hat{\rho}},
 \end{aligned} \tag{75}$$

where the other fields are invariant and where the capital fields are type IIA and the lower-case fields heterotic. To derive this string/string duality rule one must also use the two dualization formulae (48) and (56). Note that one may only derive a string/string duality rule for \hat{H} and not \hat{B} . This is of course related to the fact that from the 6-dimensional point of view the string/string duality is a symmetry of the equations of motion only.

- (B) The only other $D = 6$ duality that is purely S , and hence can be written in a 6-dimensional covariant way,²⁰ is the one that acts within the type IIB theory. We find that this is given by a particular $O(5, 21)$ transformation with parameter Ω given by

$$\Omega = \mathcal{L}, \quad (76)$$

where \mathcal{L} is the flat $O(5, 21)$ metric given in Eq. (40). In components its action on the antisymmetric tensors and scalars is given by

$$\begin{aligned} \hat{H}'^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\ \hat{H}'^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\ \hat{H}'^{ta}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= (L\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}})^a, \\ e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}} \left(e^{-4\hat{\phi}} - e^{-2\hat{\phi}'} \hat{\ell}^a \hat{\ell}^b \hat{M}_{ab}^{-1} + \frac{1}{4} \ell^4 \right)^{-1}, \\ \hat{\ell}^a &= \frac{e^{-2\hat{\phi}} \hat{\ell}^c \hat{M}_{cd}^{-1} L^{da} - \frac{1}{2} \hat{\ell}^2 \hat{\ell}^a}{e^{-4\hat{\phi}} - e^{-2\hat{\phi}} \hat{\ell}^a \hat{\ell}^b \hat{M}_{ab}^{-1} + \frac{1}{4} \hat{\ell}^4}, \\ \hat{M} &= \hat{M}^{-1}. \end{aligned} \quad (77)$$

Note that, when restricted to the common subsector, this duality transformation indeed reduces to the standard S -duality rule given in Section 2.

- (C) We finally give an example of a discrete duality that involves a T -duality and hence cannot be written in a 6-dimensional covariant. The example we consider concerns the T -duality map from the type IIA onto the type IIB theory. Following our method described above we find the following expression for this duality transformation:

$$\begin{aligned} \hat{\Phi} &= \hat{\phi} - \frac{1}{2} \log(-\hat{g}_{xx}), \\ \hat{G}_{xx} &= 1/\hat{g}_{xx}, \\ \hat{G}_{x\mu} &= \hat{b}_{x\mu}/\hat{g}_{xx}, \\ \hat{G}_{\mu\nu} &= \hat{g}_{\mu\nu} - \left(\hat{g}_{x\mu} \hat{g}_{x\nu} - \hat{b}_{x\mu} \hat{b}_{x\nu} \right) / \hat{g}_{xx}, \\ \hat{B}_{x\mu}^{(1)} &= \hat{g}_{x\mu} / \hat{g}_{xx}, \\ \hat{B}_{\mu\nu}^{(1)} &= \hat{b}_{\mu\nu} - \left(\hat{g}_{x\mu} \hat{b}_{x\nu} - \hat{g}_{x\nu} \hat{b}_{x\mu} \right) / \hat{g}_{xx}, \\ \hat{B}_{x\mu}^a &= \hat{v}_\mu^a - \hat{v}_x^a \hat{g}_{x\mu} / \hat{g}_{xx}, \\ \hat{\ell}^a &= \hat{v}_x^a, \end{aligned}$$

²⁰ The reason that the S -duality rules can be written in a 6-dimensional covariant way, i.e. in terms of the $\hat{\mu}$ -indices, is that, in contrast to the T -duality, their presence does *not* require the existence of a special isometry direction.

$$\hat{M}_{ab} = \hat{m}_{ab}, \quad (78)$$

where the capital fields are IIB and the lower-case fields are IIA fields, respectively. Note that the duality transformations of $\hat{B}_{\mu\nu}^{(2)}$ and $\hat{B}_{\mu\nu}^a$ are not given. Their transformation rules follow from the ones given above via the self-duality conditions (42).

6. The 6-dimensional chiral null model

To give the reader a better understanding of how the discrete duality symmetries constructed in this paper relate different backgrounds to each other we discuss in this section the chiral null model as a special example. We will present here only some aspects related to the common sector. A more detailed discussion, including the type II sector, will be given elsewhere [25]. Our starting point is the reduction of the $D = 6$ common sector to $D = 5$ which defines the **e** element. Next, we oxidize back to six dimensions in two different ways, related to the S^{-1} and $(TST)^{-1}$ element indicated in Fig. 6. Starting with the chiral null model, these two group elements lead to two other 6-dimensional solutions. We thus end up with three different 6-dimensional solutions which are all dual to each other. We will describe these three solutions below.

6.1. The **e** element

The chiral null model is a string background that allows one conserved chiral current on the world sheet. As a $D = 10$ heterotic string background it has unbroken supersymmetries and is exact in the α' expansion [26].²¹ The dimensional reduction yield many known black hole and Taub–Nut geometries [28,29]. Since we are interested here only in the common sector we assume a trivial reduction from $D = 10$ to $D = 6$ and thus obtain the following expression for the $D = 6$ chiral null model (in the string-frame metric):

$$\begin{aligned} d\hat{s}^2 &= 2F(x) du \left[dv - \frac{1}{2}K(x) du + \omega_I dx^I \right] - dx^I dx^I, \\ \hat{B} &= -2F(x) du \wedge [dv + \omega_I dx^I], \quad e^{2\hat{\phi}} = F(x), \end{aligned} \quad (79)$$

where u, v are standard light-cone coordinates, F^{-1} and K are harmonic functions and ω_I fulfills the Maxwell equation with respect to the transversal coordinates x^I ($I, J = 1, \dots, 4$):

$$\partial^2 F^{-1} = \partial^2 K = \partial_I \partial_{[I} \omega_{J]} = 0. \quad (80)$$

In the case of $\omega_I = 0$ this model is an interpolation between the gravitational wave background ($F = 1$) and fundamental string ($K = 1$).²²

²¹ The issue of unbroken supersymmetry and α' corrections for special cases of the $D = 10$ chiral null model has been studied in [27].

²² The standard parameterization of the fundamental string is given by $K = 0$. However, this would lead to a singularity since we have to invert K below. For this reason we make use of the possibility to give v a linear

Using the reduction formulae given in Eq. (17) we reduce this solution to five dimensions. If we assume that the internal direction is u (note: $\hat{g}_{xx} = \hat{g}_{uu} < 0$, i.e. space-like) and v is the time we find for the fields in the Einstein frame

$$\begin{aligned} ds^2 &= \left(\frac{F}{K}\right)^{2/3} (dt + \omega_I dx^I)^2 - \left(\frac{K}{F}\right)^{1/3} dx_I dx_I, & H_{\mu\nu\rho} &= 2FK A_{[\mu} \partial_\nu A_{\rho]}, \\ e^{-\frac{4}{\sqrt{3}}\sigma} &= FK, & e^{-4\phi} &= \frac{K}{F}, \\ A_\mu &= -K^{-1}(1, \omega_I), & B_\mu &= -F(1, \omega_I). \end{aligned} \quad (81)$$

These fields define a solution of the 5-dimensional type II theory. The 6-dimensional solution related to the \mathbf{e} element is given by the original $D = 6$ chiral null model given in (79). To construct the other dual solutions we have to oxidize the $D = 5$ solution in different ways. Before we can do that we have to dualize the torsion which defines the third vector C_μ . The result (in Einstein-frame) is

$$\begin{aligned} F_{\sigma\tau}(C) &= \partial_\sigma C_\tau - \partial_\tau C_\sigma = \frac{1}{2} \frac{1}{\sqrt{g}} e^{-\frac{8}{3}\phi} g_{\sigma\alpha} g_{\tau\beta} \epsilon^{\alpha\beta\mu\nu\rho} H_{\mu\nu\rho} \\ &= \frac{1}{2} \delta_{\sigma I} \delta_{\tau J} \epsilon^{IJKL} \partial_K \omega_L, \end{aligned} \quad (82)$$

where we have used Eq. (48). To make the situation more transparent we restrict ourselves to the static limit (diagonal metric): $\omega_I = 0$, which has the consequence that

$$C_\mu = 0. \quad (83)$$

6.2. The S^{-1} element

This element was related to the field redefinition (see (28) and (31))

$$\sigma' = \frac{1}{2}(\sigma + \sqrt{3}\phi), \quad \phi' = \frac{1}{2}(\sqrt{3}\sigma - \phi) \quad \text{and} \quad B \leftrightarrow C. \quad (84)$$

We have to insert for σ and ϕ the functions given in (81) and take σ' for the new compactification radius and ϕ' for the new dilaton. Furthermore, B was the KK field coming from the antisymmetric tensor and the interchange of the two gauge fields means that we have no KK gauge field coming from the torsion but we have a non-vanishing $D = 5$ torsion. The new $D = 6$ solution is then given by (using the string-frame metric)

$$\begin{aligned} d\hat{s}^2 &= 2du \left(dv - \frac{1}{2}K du \right) - F^{-1} dx_I dx_I, & e^{-2\hat{\phi}} &= F, \\ \hat{H}_{IJK} &= \frac{1}{6} \epsilon_{IJKL} \partial^L F^{-1}, & \hat{H}_{IJu} &= \hat{H}_{IJu} = \hat{H}_{Iuv} = 0. \end{aligned} \quad (85)$$

This solution has been discussed already in [30] and the generalization to stationary metrics ($\omega \neq 0$, i.e. non-vanishing angular momentum in $D = 5$) can be found in [31]. It can be interpreted as an interpolating solution between a wave background ($F = 1$) and the solitonic string ($K = 1$) solution [5,8,9].

shift. We thus obtain $K = 1$ as a fundamental string. Physically this means that we give the string a linear momentum (see Ref. [26]).

6.3. The $(\mathbf{TST})^{-1}$ element

For this element we have the field redefinition (see (28) and (31))

$$\sigma' = \frac{1}{2}(\sigma - \sqrt{3}\phi), \quad \phi' = -\frac{1}{2}(\sqrt{3}\sigma + \phi) \quad \text{and} \quad A \leftrightarrow C. \quad (86)$$

Since we are considering $C = 0$ only, the interchange of the vectors means here, that we have no KK gauge field coming from the metric (no off-diagonal term). We find for the 6-dimensional solution (in the string-frame metric)

$$\begin{aligned} d\hat{s}^2 &= F (dv^2 - du^2) - K dx_I dx_I, & e^{2\hat{\phi}} &= FK, \\ \hat{H}_{IJK} &= -\frac{1}{6}\epsilon_{IJKL}\partial^L K, & \hat{H}_{Iuv} &= \partial_I F, & \hat{H}_{IJu} &= \hat{H}_{IJv} = 0. \end{aligned} \quad (87)$$

This string type solution was discussed before in [32]. In analogy to the previous cases we can now interpret this solution as an interpolating solution between the fundamental string ($F = 1$) and solitonic string ($K = 1$). Note that in relation to the \mathbf{e} element (the original solution (79)) the dilaton and the compactification radius (\hat{g}_{uu}) have interchange their role.

Finally, one can ask what about the other cases: \mathbf{T} , \mathbf{TS} or \mathbf{ST} ? From the previous sections we know that these cases correspond to the 5-dimensional solution that is T -dual to our 5-dimensional starting point (81). From (28) and (31) we find the replacements

$$\sigma' = -\sigma, \quad \phi' = \phi \quad \text{and} \quad A \leftrightarrow B, \quad (88)$$

which simply means that we have to interchange both harmonic functions

$$K \leftrightarrow F^{-1}. \quad (89)$$

This, however, does not change the structure of the 6-dimensional solutions. Thus, these elements are related to internal symmetries of every 6-dimensional solution whereas the other elements (\mathbf{S} and \mathbf{TST}) correspond to solution generating transformations.

Inserting now harmonic functions for K and F^{-1} we first find that all three solutions are asymptotically free and that near the singularity the \mathbf{e} solution is in the weak coupling region, the \mathbf{S} solution is strongly coupled whereas for the \mathbf{TST} solution the string coupling constant remains finite. In a forthcoming paper [25] we will discuss the charge and mass spectrum of these solutions. Also, we will add 6-dimensional gauge fields and generalize them to truly heterotic (79), type IIA (85) and type IIB (87) solutions with non-trivial RR fields.

Acknowledgements

We would like to thank M. Bianchi, H.J. Boonstra, G. Papadopoulos, A. Sagnotti, P. Townsend and A. Tseytlin for discussions. In particular we would like to thank Tomas Ortín who was involved in the initial discussions leading to the present work. One of us (K.B.) would like to thank Groningen University for hospitality. The work of K.B.

is supported by a grant of the DFG. The work of E.B. has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW). The work of B.J. was performed as part of the research programme of the “Stichting voor Fundamenteel Onderzoek der Materie” (FOM).

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